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Research on this contract was directed towards areas of mathematics and numerical computation which have applications to image/signal processing. The research can be broadly classified into the following areas: (i) image processing, (ii) processing digital terrain data, (iii) wavelets and multiresolution, (iv) information theory, (v) analogue to digital signal conversion, and (vi) approximation. The highlights of the scientific accomplishments include: development of compression methods for Digital Terrain Elevation Data, solution of extremal problems for image denoising using wavelets, construction of new wavelet bases, development of high order methods for conversion between analogue and digital signals.

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## HIGHLY NONLINEAR ALGORITHMS FOR WAVELET BASED IMAGE PROCESSING WITH MILITARY APPLICATIONS

ARO Contract DAAG55-98-1-0002

Principal Investigators: R. DeVore and P. Petrushev

**Overview:** Research on this contract was directed towards areas of mathematics and numerical computation which have applications to image/signal processing. The research can be broadly classified into the following areas: (i) image processing, (ii) wavelets and multiresolution, (iii) information theory, (iv) analogue to digital signal conversion, and (v) approximation. We shall now briefly summarize the highlights of our work in these areas.

### Image Processing

We have concentrated on techniques which are important for automated target recognition. The first step is usually to compress and remove noise from the image. Later steps include feature extraction and image registration. Some highlights of our work are the following:

- **Noise suppression.**

The space BV of functions of bounded variation is often used as a model for real images (an idea of Mumford and Shah). Therefore this space appears in important extremal problems resting at the core of noise reduction in images. In [CDPX], we make the **remarkable discovery** that wavelet thresholding is a minimizer for the K-functional between  $L_2$  and BV. Heretofore, this minimization problem was solved only using PDE techniques (see e.g. [LOR]) which is numerically intensive. The proof that wavelet techniques also minimize is deep and depends on a new adaptive algorithm for piecewise constant approximation. The numerical implementation of these results give visually comparable performance to the LOR algorithms with far less computation.

An important consequence of our work in [CDPX] is the proof that a function in BV has wavelet coefficients in weak  $\ell_1$ . This results has had considerable had resonance in PDEs. Yves Meyer has shown that our results give new embedding inequalities (Poincaré, Gagliardo-Nirenberg) for Besov spaces which are important in PDEs. He then conjectured other improvements which could not be derived from the weak  $\ell_1$  results. In [CDDD1] we prove the conjectures of Meyer by giving a finer harmonic analysis for BV consisting in a scale of weighted weak  $\ell_1$  theorems for the wavelet coefficients of BV functions.

In [CDKP], we analyze thresholding algorithms used in statistical estimation and show that to each such algorithm one can associate a certain maximal space which benchmarks their performance. The main tool for obtaining maximal spaces is to develop an approximation theory [CDH] which describes error rates when the error is measured in  $L_p$  (not necessarily  $p = 2$ ) while the thresholding is done relative to another  $L_r$  norm.

- Greedy algorithms and adaptive pursuit.

Adaptive pursuit from a dictionary of functions is used frequently in feature extraction. We are continuing the work in [DT,DT1] in trying to understand the theoretical advantages and disadvantages of these highly nonlinear methods. We have given the first estimates for approximation by greedy algorithms and studied the improvements obtained by various modifications of greedy algorithms. This is the first semblance of a coherent theory for highly nonlinear approximation. **The paper [DT1] was selected the best paper of 1997 by the Journal of Complexity Theory.** Our subsequent work is directed at showing unconditional democratic bases are best bases for approximation and encoding.

### Digital Terrain Elevation Data

Digital Terrain Elevation Data (DTED) maps are now available for most of the earth's surface. At present, data of varying resolution are available: Level 0 (1 km.), Level 1 (100 m.), and Level 2 (10 m. resolution). These data sets present a rich resource of multiple resolution data for both military and commercial applications. However, the effective utilization of these data sets requires new processing techniques for rapid access, query, and display of this data. Moreover, the typical utilization of DTED maps will be in conjunction with data of other modalities such as sensor data (FLIR, IR, SAR, etc.), cultural and thematic data (such as power lines, man-made structures, roadways, geopolitical boundaries), and additional textured terrain features (landmarks, rivers, marshes, soil properties). The integration of all of these data sources into one coherent resource will require the registration of data sets of different modalities.

DTED maps are usually rendered as 3-D surfaces and image processing techniques may not be appropriate for processing these maps. For example, in many applications, distortion in elevation is more important than the mean square deviation as is typically used in image processing. Thus, the metrics used to measure distortion in such applications as compression, noise reduction, and registration for DTED maps should be with respect to the maximum deviation norm or Hausdorff metric.

USC has assembled a team (led by the PIs) of research faculty, postdoctoral, graduate and undergraduate students who are working to develop multiscale methods for the rapid processing of large data sets with a particular emphasis on the application of this processing platform to DTED maps and associated imagery. Several algorithms are currently under investigation for efficient and accurate processing of DTED maps for registration, denoising, compression, and rendering operations.

Some of the desired features of the algorithms under development are: (a) high compression, (b) robust error handling, (d) progressive transmission of the data, (c) quick rendering, and (e) burning in (tunneling), and (f) line of sight display. Since almost all graphic hardware uses triangular polygonal patches as building blocks for object description, we have focused our attention on algorithms utilizing meshes of polygonal elements. Two classes of algorithms are being investigated:

- (i) Nonlinear approximation algorithms based on adaptive multiresolution analysis,
- (ii) Greedy (insertion or removal) algorithms for mesh construction which utilize Delaunay triangulation.

The first algorithms include: (a) initial coarse adaptive triangulation which allows a low resolution good approximation, (b) wavelet decomposition of the function for achieving sparse representation of the function (surface), (c) conversion to hierarchical B-spline representation and application of the nonlinear uniform approximation schemes adapted from [DJL] and [DPY], (d) compression and progressive transmission of the data using the hierarchical representation.

The greedy removal algorithm is a recursive procedure with the following basic elements: (a) determination and upgrading of the significance table of the grid points, (b) removal one by one of the least significant points, (c) mesh updating after each removal with Delaunay triangulation algorithm. The greedy insertion algorithm utilizes the same elements but in a reverse order. We pay special attention to the data structures that enable us to compress and transmit the data progressively.

We develop both the theoretical foundations and the practical aspects of the methods. In [KP], we develop the theory of nonlinear  $n$ -term approximation from piecewise linear Courant elements or (discontinuous) piecewise polynomials generated by hierarchical nested triangulations of  $R^2$ . This allows for the possibility of sharp angles in the triangulations which is a desirable feature for DTED which is not allowed in other developments of finite element approximation. To characterize the rate of approximation we introduce and develop three families of smoothness spaces generated by nested triangulations. We call them B-spaces because they can be viewed as variations of Besov spaces and are natural for the appropriate Jackson and Bernstein estimates for  $n$ -term approximation generated by an arbitrary nested triangulation and, consequently, enable characterization of the corresponding approximation spaces.

Although we have developed methods for  $n$ -term piecewise polynomial approximation which provide the best *asymptotic* rates of approximation, practical implementation issues will ultimately determine the extent to which these methods are useful in applications. We have formed an algorithm implementation team to develop research software incorporating both the nonlinear approximation and greedy methods mentioned earlier. There is a common modularized framework (OPTA) for possible variations of the algorithms which will provide the flexibility of designing and testing proposed hybrid methods, and to will allow us to compare our results with those of other researchers. The various components are built upon a common elemental base of hierarchical structures naturally suited for multiresolution analysis, data operations, and rendering of the surfaces. Specialized data structures have been tailored to take advantage of particular attributes of the approximation algorithms under investigation. In order to test the composite approximation and rendering schemes under a variety of computing environments, code development is performed in C, using the GLUT libraries for windowing and OpenGL language for graphics rendering. Special consideration has been given to insuring that rendering algorithms take full advantage of hardware available on a given platform. Additional attention in the development is being paid to future options of distributed and massively parallel development, since the algorithms are inherently scalable.

The potential applications of this technology include mission planning and rehearsal, autonomous navigation, rapid registration of imagery to DTED, post battle assessment, rapid telecommunication, and efficient touring, querying and storage of large data bases.

### Wavelets and multi-resolution

We continue to develop the theoretical aspects of wavelets and multiresolution. We have two main goals: (i) to extend the usual theory to be able to treat more general domains such as occur in PDEs and image processing, (ii) to construct a variety of unconditional bases for function spaces which can then be utilized in numerical algorithms.

- **Multiresolution on domains.** Cohen, Dahmen and DeVore [CDD] develop multiresolution on domains with the aim of treating PDEs on domains. However, applications to image processing are also clear. We show how to construct from the usual wavelets a multiresolution on Lipschitz domains. This is the analogue of wavelets on an interval although the constructions in the multidimensional case are much more subtle.

- **Shift invariant spaces.** In a series of papers, de Boor, DeVore, and Ron [BDR1-4] have developed the structural properties of shift invariant spaces and have shown how these properties can be used in wavelet analysis. The most important of their results is the complete characterization of approximation order for dilates of shift invariant spaces. In [BDR4], we give simpler ways to implement our theory in the multiresolution setting. For example, the results of [BDR4] can be applied readily to multiwavelets.

- **Construction of unconditional bases.** The construction of unconditional bases for a given function space is valuable in many applications. Such bases frequently lead to a simple characterization of this space in terms of norms applied to the sequence of coefficients with respect to that basis. The sequence norm characterization then permits the solution of extremal problems. Unconditional bases in this context have been used in many fields such as statistics [Do], image processing [DJKP], nonlinear approximation [De], and functional analysis (for example the characterization of K-functionals). From many perspectives, it is of great benefit if one can prescribe the nature of these bases. For example, wavelet bases are popular because of their time-frequency localization. We are interested therefore in constructing bases in which the character of the basis functions is prescribed in advance. In [P1] and [KP1], we put forward a method which allows this flexibility. In [P1], a basis consisting of rational functions of bounded degrees is constructed. This solves a long standing problem concerning the existence of such bases. In [KP1], we extend the construction to multivariate functions. We also establish that these bases are unconditional for a wide range of function spaces, namely any spaces from the classes of Triebel-Lizorkin or Besov spaces. The new bases are utilized to nonlinear approximation.



## Analogue to digital (A/D) converters

There is great demand for efficient analogue to digital converters in both the civilian and military sectors. State of the art methods are the sigma-delta converters which consist of three steps: (i) oversampling the signal, (ii) one bit quantization of the oversampled values, (iii) filter reconstruction. Sigma-delta converters work well but there was no mathematical understanding of why. Daubechies and DeVore [DD] began to mathematically analyze sigma-delta converters and have been able to quantify their success. They have also derived higher order methods which theoretically perform better than existing converters. These methods are being analyzed further for their practicality and implementation in silicon. There is presently a small consortium consisting of Daubechies, DeVore, and researchers at AT&T, Lucent, and Hewlett Packard, who are investigating the performance of the new sigma-delta architectures. Several research papers announcing these results are being prepared.

## Neural Networks

Neural networks are frequently used in statistical decision theory, classification, and feature extraction. Our goal in this research is to understand their advantages over more traditional numerical methods. In [DOP], DeVore, Oskolkov, and Petrushev prove the first tight theorems on linear approximation using neural networks and discover the interesting fact that the unit impulse function gives a feedforward network with efficiency  $\mathcal{O}(n^{-3/2})$  in two dimensions. The expected result is  $\mathcal{O}(n^{-1})$  since one is approximating by piecewise constants. These results were generalized to arbitrary space dimension by Petrushev [P]. The results of DeVore and Temlyakov [DT] on greedy algorithms can also be used for nonlinear approximation by neural networks.

## Information Theory

In [DVDD], we have given a survey and state of the field accounting of the role of Harmonic Analysis in information theory, especially data compression. The main point of this article is that advances in information theory parallel advances in harmonic analysis. Also harmonic analysis gives the theoretical underpinnings which explain the connections between the Shannon (stochastic) theory and the Kolmogorov (deterministic) theory.

In [CDDD], we introduce universal encoding algorithms based on wavelet decompositions and organization of these decompositions along trees. We prove that appropriate Besov regularity of order  $s$  guarantees error bounds of the order  $\mathcal{O}(N^{-s/d})$  where  $N$  is the number of bits used in the encoding. This theory complements the theory for compression of images given in [DJL] which only took into account coefficient count (not bit count). We are now analyzing the performance of these tree based encoders.

A follow up to the work in [CDDD] is the paper [BDKY] which gives an algorithm for finding best tree approximations and then characterizes (through the use of maximal

operators) the functions which possess a given rate of tree approximation.

### Approximation Theory

Many problems in applied mathematics can be viewed from approximation theory. The most obvious are numerical computation where numerical efficiency is related to rate of approximation. Approximation theory also can be used in image processing, regularity for PDEs, and statistical estimation. Here are some highlights of our work in approximation.

- **Wavelet approximation.** DeVore, Kyriazis, and Wang [DKW] have analyzed the approximation of multiresolution on general domains. The goal of this research (which has to a large extent been accomplished) is to extend the usual approximation results (which include nonlinear approximation) to general domains. This requires approximation in the spaces  $L_p$ ,  $0 < p < 1$ .

Cohen, DeVore, and Hochmuth [CDH] characterize the rate of approximation for certain types of thresholding that occurs in statistical estimation. The thresholding is done in  $L_r$  while the approximation efficiency is measured in  $L_p$ . We relate these questions to interpolation of Besov spaces for certain (non-traditional) parameters.

We have introduced in [CDDD] the concept of tree approximation which emulates what is used in image encoding. In [BDKY], we show how to use certain maximal functions defined on trees to find best tree approximants. We also characterize the functions which can be approximated with a prescribed rate using tree approximation.

We introduce in [P2] a family of new spaces (B-spaces) which are associated with anisotropic dyadic partitions of  $R^d$ . We use the B-spaces to characterize non-linear approximation from piece-wise polynomials over dyadic partitions. We prove an estimate that relates the multivariate rational approximation to piece-wise polynomial approximation in  $L_p$  ( $0 < p < \infty$ ) which we utilize to estimate the rational approximation in terms of the minimal B-norm.

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## Publications from Contract DAAG55-98-1-0002

### PAPERS IN PRINT

- A. Cohen, R. DeVore, and R. Hochmuth, *Restricted nonlinear approximation*, Constructive Approximation **16** (2000), 85-113.
- R. DeVore, *Adaptive Numerical Methods for PDES*, ESAIM: Proceedings Actes du 32EME Congres D'Analyse Numerique, Canum (Sept. 2000).
- D. Donoho, M. Vetterli, R. DeVore, and I. Daubechies, *Data compression and harmonic analysis*, IEEE Trans. Information Theory **44** (1998), 2435-2476.
- P. Petrushev, *Bases consisting of rational functions of uniformly bounded degree or more general functions*, J. of Funct. Anal. **174** (2000), 18-75.

### PAPERS IN PRESS

- A. Cohen, R. DeVore, G. Kerkyacharian, and D. Picard, *Maximal spaces with given rate of convergence for thresholding algorithms*, ACHA.
- A. Cohen, W. Dahmen, I. Daubechies, and R. DeVore, *Tree approximation and encoding*, ACHA.

### PAPERS SUBMITTED

- A. Cohen, W. Dahmen, I. Daubechies, and R. DeVore, *Harmonic analysis of the space BV*.
- R. Baraniuk, R. DeVore, G. Kyriazis, and X. M. Yu, *Best tree approximation*.
- B. Karaivanov, P. Petrushev, *Multivariate piece-wise polynomial approximation beyond Besov spaces*.
- G. Kyriazis, P. Petrushev, *New bases for Triebel-Lizorkin and Besov spaces*.
- P. Petrushev, *B-spaces, anisotropic Haar bases, and multivariate rational approximation*.

### PAPERS IN PREPARATION

- I. Daubechies, R. DeVore, *Oversampling and coarse bit quantization*.

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